On the propagation of a perturbation in an anharmonic system

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Abstract

We give a not trivial upper bound on the velocity of disturbances in an infinitely extended anharmonic system at thermal equilibrium. The proof is achieved by combining a control on the non equilibrium dynamics with an explicit use of the state invariance with respect to the time evolution.

Key words: Anharmonic crystals, propagation velocity.

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1 Introduction

In the present paper we investigate the long time behavior of an infinitely extended anharmonic system, that represents, of course, a schematic model of a crystal. There are several papers devoted to the time evolution of systems containing an infinite number of components, either particles moving in a continuum or lattice systems, see Refs. [2, 5, 8–15, 17, 20, 21, 23], but few of them study the asymptotic (in time) behavior of the system. The reason lies in the difficulty to obtain dynamical estimates that remain good for very long times. Some results in this direction have been recently obtained [3, 4, 6, 7], but they are not related to the topic of the present paper.

Such dynamical estimates can be greatly improved if the physical system is assumed at thermal equilibrium. Actually, it is not necessary to consider exactly a Gibbs state: the present results apply to any reasonable time invariant (i.e. stationary) state. This assumption allows to prove the locality in space of the motion and hence its existence, see Refs. [1, 16, 19,

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24, 25, 27–29]. Moreover, in this case it is known (but perhaps not sufficiently underlined) that the magnitude of a component of the system may increase only very slowly in time. This fact allows us to obtain not trivial results for the following problem.

Consider a lattice system in dimension d, with anharmonic oscillators at each site. At time zero we perturb the oscillator located on the site i and study the influence at time t on that located on the site j. For harmonic oscillators or bounded rotators it has been proved in Ref. [22] that this influence becomes negligible when $|i - j| > ct$, for some c which thus gives a bound of the velocity of propagation of the perturbation.

The case of anharmonic oscillators is much more involved. The general case seems too difficult, while for systems in thermal equilibrium a reasonable estimate has been obtained in Ref. [22]: the influence of the perturbation is exponentially small in time for distances $|i - j| > t^{4/3}$.

At this point we must precise what we mean by perturbation. Actually, we do not modify the equilibrium state, rather we analyze observables which give an estimate of the size of the correlation. Following Ref. [22], we use the Poisson brackets to this purpose, by analogy with a similar problem in the quantum case previously studied in Ref. [18]. More precisely, we analyze the Poisson brackets $\{f_i, g_j \circ \Phi_t\}$ where f_i, g_j are observables localized on the site i, j respectively, and Φ_t denotes the time evolution. We prove that, with probability one with respect to any reasonable stationary state, $\{f_i, g_j \circ \Phi_t\}$ is exponentially small in time whenever $|i - j| > t \log^\alpha t$ for a suitable $\alpha > 0$. This gives a non trivial bound on the velocity of the perturbation.

The problems and the results are rigorously posed in Section 2 and proved in Section 3.

2 Notation and statement of the result

At each point i of the d-dimensional lattice $\mathbb{Z}^d$ there is an oscillator with coordinate $q_i \in \mathbb{R}$ and momentum $p_i \in \mathbb{R}$. The state of the system is thus determined by the infinite sequence $x = \{x_i\}_{i \in \mathbb{Z}^d} = \{(q_i, p_i)\}_{i \in \mathbb{Z}^d}$ of positions and momenta of the oscillators. We shall denote by $\mathcal{X}$ the set of all such possible states, equipped with the product topology.

The time evolution $t \mapsto x(t) = \{(q_i(t), p_i(t))\}_{i \in \mathbb{Z}^d}$ is defined by the solutions of the following infinite set of coupled differential equations,

$$\begin{cases} \dot{q}_i(t) = p_i(t) \\ \dot{p}_i(t) = F_i(x(t)) \end{cases} \qquad i \in \mathbb{Z}^d, \qquad (2.1)$$

where the force $F_i(x)$ induced by the configuration $x = \{(q_i, p_i)\}_{i \in \mathbb{Z}^d}$ on

the i-th oscillator is given by

$$F_i(x) = -U'(q_i) - K \sum_{j:|j-i|=1} (q_i - q_j), \qquad (2.2)$$

with $K > 0$ and $U(\cdot)$ a non negative polynomial of degree 4 with strictly positive leading coefficient. Here $|i-j|$ is the distance between the points $i = (i_1, \ldots, i_d)$ and $j = (j_1, \ldots, j_d)$ defined by

$$|i-j| = \sum_{\ell=1}^{d} |i_\ell - j_\ell|.$$

In order to consider configurations that are typical for any reasonable thermodynamic equilibrium state, we allow initial data with logarithmic divergences in the energy. More precisely, for $\nu \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, let

$$W_{\nu,k}(x) \doteq \sum_{i \in \Lambda_{\nu,k}} \left\{ \frac{p_i^2}{2} + U(q_i) + 1 \right\} + \sum_{\substack{i,j \in \Lambda_{\nu,k} \\ |j-i|=1}} \frac{K}{4}(q_i - q_j)^2, \qquad (2.3)$$

where $\Lambda_{\nu,k}$ denotes the cube of center ν and side $2k+1$. By defining

$$Q(x) \doteq \sup_{\nu \in \mathbb{Z}^d} \sup_{k > \log^{1/d}(e+|\nu|)} \frac{W_{\nu,k}(x)}{(2k+1)^d}, \qquad (2.4)$$

we denote by $\mathcal{X}_0$ the following subset of $\mathcal{X}$,

$$\mathcal{X}_0 = \{x \in \mathcal{X} : Q(x) < \infty\}.$$

Let now ω be any Borel probability measure on $\mathcal{X}$ that satisfies the following superstability estimate: there exists a positive constant C_ω such that, for any λ small enough,

$$\omega\left(e^{\lambda W_{\nu,k}}\right) \leqslant e^{C_\omega (2k+1)^d} \qquad \forall \nu \in \mathbb{Z}^d \quad \forall k \in \mathbb{N}. \qquad (2.5)$$

It is proved in the Appendix that this implies, for any λ small enough,

$$\lim_{N \to +\infty} e^{\lambda N} \omega(Q > N) = 0, \qquad (2.6)$$

which in particular yields $\omega(\mathcal{X}_0) = 1$.

The following theorem gives existence and uniqueness of the solution to Eq. (2.1) for initial data in the set $\mathcal{X}_0$.

Theorem 2.1 There exists a one-parameter group of transformations $\Phi_t : \mathcal{X}_0 \to \mathcal{X}_0$, $t \in \mathbb{R}$, such that $t \to \Phi_t(x)$ is the unique global solution to Eq. (2.1) with initial condition $\Phi_0(x) = x$. Moreover $\Phi_t(x)_j = x_j(t) = (q_j(t), p_j(t))$ are differentiable functions with respect to $x_i = (q_i, p_i)$ for any $j, i \in \mathbb{Z}^d$ and $t \in \mathbb{R}$. Finally, there exists a constant C_0 such that

$$Q(\Phi_t(x)) \leqslant C_0 \Big\{ Q(x) \log[e + Q(x)] + t^4 \Big\}. \tag{2.7}$$

The proof of Theorem 2.1 is given in the Appendix. We remark that a global existence and uniqueness theorem for a larger class of initial data can be found in Ref. [17]. Unfortunately, the proof given there does not guarantee that the set $\mathcal{X}_0$ is invariant under the dynamics. We instead adapt to the present context the technique developed by Dobrushin and Fritz in the case of particles in the continuum [11, 13].

Let $\mathcal{U}$ be the algebra of all local observables. Thus $f : \mathcal{X} \to \mathbb{R}$ is an element of $\mathcal{U}$ iff $f(x) = f_\Lambda(x_\Lambda)$ for some bounded set $\Lambda \subset \mathbb{Z}^d$ and some differentiable function f_Λ, depending on the finite set of real variables $x_\Lambda = \{x_i\}_{i \in \Lambda}$, which is bounded with its derivatives. The action of the time evolution on the local observables is still denoted by Φ_t: if $f \in \mathcal{U}$ the function $\Phi_t f$ is defined by setting $\Phi_t f(x) = f(\Phi_t(x))$ for any $x \in \mathcal{X}_0$. We finally observe that the Poisson brackets

$$\{f, g\}(x) = \sum_{i \in \mathbb{Z}^d} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)(x)$$

are well defined for any $f, g \in \mathcal{U}$.

A time invariant state ω is a probability measure on $\mathcal{X}$ for which (2.5) holds and such that $\omega(\Phi_t f) = \omega(f)$ for any $f \in \mathcal{U}$ and $t \in \mathbb{R}$. The typical example of such a state is given by the infinite Gibbs measure obtained as the thermodynamic limit with free boundary conditions [26].

Given a pair of differentiable function $f, g : \mathbb{R}^2 \to \mathbb{R}$, which are bounded with their derivatives, we denote by f_i, resp. g_j, the local observable defined by setting $f_i(x) = f(x_i)$, resp. $g_j(x) = g(x_j)$. Our main result concerns the asymptotic behavior of the Poisson brackets $\{f_i, \Phi_t g_j\}(x)$ in the limit when simultaneously $|i - j| \to \infty$ and $t \to \infty$.

Theorem 2.2 Let ω be any time invariant state satisfying (2.5). Then for each f, g as above, $\alpha > 1/2$, and $b > 0$ we have

$$\lim_{t \to \infty} \sup_{j \,:\, |i-j| > t \log^\alpha t} e^{bt} \{f_i, \Phi_t g_j\}(x) = 0, \tag{2.8}$$

almost surely with respect to the probability measure ω.

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Remark. We consider the case of one dimensional oscillators with quartic one-body potential and nearest-neighbor harmonic interaction only for the sake of simplicity. Indeed, the same strategy applies to the general case of N-dimensional oscillators, say $q_i \in \mathbb{R}^N$, such that the force $F_i(x)$ induced by the configuration $x = \{(q_i, p_i)\}_{i \in \mathbb{Z}^d}$ on the i-th oscillator is now given by

$$F_i(x) = -\nabla_{q_i} U(q_i) - \sum_{j:|j-i| \leqslant r} \nabla_{q_i} V(q_i - q_j).$$

Here r is a positive parameter, while U and V are smooth functions such that $U(\xi) = P_1(|\xi|)$ and $V(\xi) = P_2(|\xi|)$, where P_1 and P_2 are non negative polynomials with maximum degree respectively 2γ and $\leqslant \gamma$. The estimate (2.8) is now valid for any $\alpha > (\gamma - 1)/\gamma$.

We conclude the section with a notation warning: in the sequel, if not further specified, we shall denote by C a generic positive constant whose numerical value may change from line to line and it may possibly depend only on the coupling constant K and the one body interaction $U(\cdot)$.

3 Proof of Theorem 2.2

In this section we prove Theorem 2.2. We remark that the time invariance and the assumption (2.5) are the only hypothesis about the state ω used in the proof. Hence, even if ω is not a spatially homogeneous state, we can assume $i = 0$ without loss of generality (otherwise stated, the rate of convergence in (2.8) turns out to be estimated uniformly with respect to the location of the site i).

Let us briefly outline the strategy of the proof. In the case of anharmonic systems, the unboundedness of the Lipschitz constant of the force is the source of troubles in obtaining not trivial dynamical estimates. To overcome this problem, we consider a "good set" of initial data, the set $\mathcal{B}$ defined below, where the dynamics is under control for all sufficiently large times $k \in \mathbb{N}$. By (2.6) and the time invariance of ω, it is readily seen that $\omega(\mathcal{B}) = 1$. On the other hand, using rough a priori bounds on the dynamics for short times, an obvious interpolation shows that in the set $\mathcal{B}$ the time evolution Φ_t is under control for any $t \geqslant 0$. Such a control turns out to be good enough to solve and bound, by iteration, the (linear) variational equation for the disturbance. We now proceed with the proof.

Given $\alpha > 1/2$ as in the statement of Theorem 2.2, we choose $\delta \in (1, 4\alpha - 1)$ and define

$$\mathcal{B} \doteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathcal{B}_k, \qquad \mathcal{B}_k \doteq \big\{ x \in \mathcal{X} \; : \; Q(\Phi_k(x)) \leqslant \log^{\delta} k \big\}.$$

Setting $\mathcal{B}_k^{\complement} \doteq \mathcal{X} \setminus \mathcal{B}_k$, since ω is time invariant, we have

$$\omega\big(\mathcal{B}_k^{\complement}\big) = \omega\big(Q \circ \Phi_k > \log^\delta k\big) = \omega\big(Q > \log^\delta k\big).$$

It follows, by (2.6), that if λ is small enough then $e^{\lambda \log^\delta k} \omega\big(\mathcal{B}_k^{\complement}\big) \to 0$ as $k \to \infty$. In particular, since $\delta > 1$, $\sum_k \omega\big(\mathcal{B}_k^{\complement}\big) < \infty$, whence

$$\omega(\mathcal{B}) = 1 \tag{3.1}$$

by the Borel-Cantelli lemma.

We next estimate:

$$
\begin{aligned}
\big|\{f_0, \Phi_t g_j\}\big|(x) &= \left|\frac{\partial f_0}{\partial q_0}\frac{\partial(\Phi_t g_j)}{\partial p_0} - \frac{\partial f_0}{\partial p_0}\frac{\partial(\Phi_t g_j)}{\partial q_0}\right|(x) \\
&= \left|\frac{\partial f_0}{\partial q_0}\left(\frac{\partial g_j}{\partial q_j}\bigg|_{\Phi_t(x)}\frac{\partial q_j(t)}{\partial p_0} + \frac{\partial g_j}{\partial p_j}\bigg|_{\Phi_t(x)}\frac{\partial p_j(t)}{\partial p_0}\right)\right. \\
&\quad \left. - \frac{\partial f_0}{\partial p_0}\left(\frac{\partial g_j}{\partial q_j}\bigg|_{\Phi_t(x)}\frac{\partial q_j(t)}{\partial q_0} + \frac{\partial g_j}{\partial p_j}\bigg|_{\Phi_t(x)}\frac{\partial p_j(t)}{\partial q_0}\right)\right|(x) \\
&\leqslant 4 \|\nabla f\|_\infty \|\nabla g\|_\infty \|\Delta_j(t, x)\|. \tag{3.2}
\end{aligned}
$$

Here $(q_j(t), p_j(t))$ denote the j-th coordinates of $\Phi_t(x)$ and $\|\Delta_j(t, x)\|$ is the uniform norm of the 2×2 Jacobian matrix $\Delta_j(t, x)$ given by

$$\Delta_j(t, x) = \frac{\partial \Phi_t(x)_j}{\partial x_0} = \begin{pmatrix} \dfrac{\partial q_j(t)}{\partial q_0} & \dfrac{\partial q_j(t)}{\partial p_0} \\[2mm] \dfrac{\partial p_j(t)}{\partial q_0} & \dfrac{\partial p_j(t)}{\partial p_0} \end{pmatrix}. \tag{3.3}$$

Hence, by (3.1) and (3.2), Theorem 2.2 follows once we show that, for any $b > 0$,

$$\lim_{t \to \infty} \sup_{j\,:\,|j|>t\log^\alpha t} e^{bt} \|\Delta_j(t, x)\| = 0 \qquad \forall x \in \mathcal{B}. \tag{3.4}$$

Lemma 3.1 There exists a positive constant $C_1 > 0$ such that, for any $x \in \mathcal{X}_0$, $j \in \mathbb{Z}^d$, and $t > 0$,

$$\|\Delta_j(t, x)\| \leqslant (1 + t) \sum_{n=|j|}^{\infty} \left(\frac{H(t, x) \log^{1/2}(e + n)}{n^2}\right)^n, \tag{3.5}$$

where

$$H(t, x) = C_1\, t^2 \left[\sup_{0 \leqslant s \leqslant t} Q(\Phi_s(x))\right]^{1/2}. \tag{3.6}$$

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Proof. By (2.1), the trajectory $(q_j(t), p_j(t)) = \Phi_t(x)_j$ satisfies the equation

$$\begin{pmatrix} q_j(t) \\ p_j(t) \end{pmatrix} = \begin{pmatrix} q_j + p_j t \\ p_j \end{pmatrix} + \int_0^t ds \begin{pmatrix} (t-s)F_j(\Phi_s(x)) \\ F_j(\Phi_s(x)) \end{pmatrix},$$

from which we get, recalling (2.2),

$$\Delta_j(t,x) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \delta_{0,j} + \sum_{h \in \mathbb{Z}^d} \int_0^t ds\, B_{j,h}(s) \begin{pmatrix} t-s & 0 \\ 1 & 0 \end{pmatrix} \Delta_h(s,x), \quad (3.7)$$

with

$$B_{j,h}(s) = -\big[U''(q_j(s)) + 2dK\big]\delta_{j,h} + K \sum_{\ell:|\ell-j|=1} \delta_{\ell,h}. \qquad (3.8)$$

The integral equation (3.7) can be solved by iteration, getting

$$\begin{aligned}
\Delta_j(t,x) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\delta_{0,j} + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, G_j(t_1,\ldots,t_n) \\
&\quad \times \begin{pmatrix} (t-t_1)(t_1-t_2)\cdots(t_{n-1}-t_n) & 0 \\ (t_1-t_2)\cdots(t_{n-1}-t_n) & 0 \end{pmatrix} \begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix},
\end{aligned} \qquad (3.9)$$

where

$$G_j(t_1,\ldots,t_n) = \begin{cases} \displaystyle\sum_{k_1,\ldots,k_n} B_{0,k_1}(t_1)\cdots B_{k_n,j}(t_n) & \text{if } |j| \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

From the definition (3.8), by using (2.3), (2.4), and recalling $U(\cdot)$ is assumed a non negative polynomial of degree 4 with strictly positive leading coefficient, it follows that

$$\big|B_{k,h}(s)\big| \leqslant C\sqrt{Q(\Phi_s(x))\log(e+|k|)} \qquad \forall\, k,h \in \mathbb{Z}^d \quad \forall\, s \in \mathbb{R}. \quad (3.10)$$

We now observe that the sum in (3.9) involves only sites k_ℓ such that $|k_\ell| \leqslant n$. On the other hand, the number of n-step walks, starting from a given site and with the possible presence of some permanences, is bounded by $(2d+1)^n$. Then, by (3.9) and (3.10),

$$\big\|\Delta_j(t,x)\big\| \leqslant (1+t) \sum_{n=|j|}^{\infty} \frac{\big[C\,t^2\,\log^{1/2}(e+n)\big]^n}{(2n)!} \Big[\sup_{0\leqslant s\leqslant t} Q(\Phi_s(x))\Big]^{n/2}.$$

By the Stirling formula the bound (3.5) now follows for a suitable choice of C_1 in (3.6). ∎

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We can now prove (3.4). Let $x \in \mathcal{B}$, by the definition of $\mathcal{B}$ there exists a positive integer $n_0 = n_0(x)$ such that $Q(\Phi_k(x)) \leqslant \log^\delta k$ for any $k \geqslant n_0$. By (2.7), for any $t > n_0$,

$$
\begin{aligned}
\sup_{0 \leqslant s \leqslant t} Q(\Phi_s(x)) \;\;\leqslant\;\; & \sup_{0 \leqslant s \leqslant n_0} Q(\Phi_s(x)) + \max_{k=n_0,\dots,[t]} \sup_{0 \leqslant s \leqslant 1} Q\big(\Phi_s\big(\Phi_k(x)\big)\big) \\
\leqslant\;\; & C_0\big\{Q(x)\log[e+Q(x)]+n_0^4\big\} \\
& + \max_{k=n_0,\dots,[t]} C_0\big\{Q\big(\Phi_k(x)\big)\log[e+Q(\Phi_k(x))]+1\big\},
\end{aligned}
$$

whence, for $t_0 = t_0(x)$ sufficiently large,

$$
\sup_{0 \leqslant s \leqslant t} Q(\Phi_s(x)) \leqslant C \log^\delta t \log\log t \qquad \forall\, t \geqslant t_0.
$$

By (3.5) and (3.6) we thus obtain, for any t large enough,

$$
\big\|\Delta_j(t,x)\big\| \leqslant (1+t) \sum_{n=|j|}^{\infty} \left(\frac{Ct^2 \sqrt{\log(e+n)\log^\delta t \log\log t}}{n^2} \right)^n .
$$

If $|j| > t\log^\alpha t$, recalling $\delta < 4\alpha - 1$, the right hand side is readily seen to be bounded by $\exp\big\{-t\log^{1/2} t\big\}$ for t sufficiently large. The limit (3.4) is thus proved.

Appendix

In this appendix we prove Eq. (2.6) and Theorem 2.1.

Proof of Eq. (2.6). From the definition (2.4) we have

$$
\omega(Q > N) \leqslant \sum_{\nu \in \mathbb{Z}^d} \sum_{k > \log^{1/d}(e+|\nu|)} \omega\big(W_{\nu,k} > N(2k+1)^d\big).
$$

Applying the Exponential Chebyshev Inequality and the assumption (2.5) on the measure ω, we next bound, for some $\lambda_0 > 0$ sufficiently small,

$$
\omega\big(W_{\nu,k} > N(2k+1)^d\big) \leqslant \omega\big(e^{\lambda_0(W_{\nu,k}-N(2k+1)^d)}\big) \leqslant e^{-(N\lambda_0 - C_\omega)(2k+1)^d}.
$$

Then, for any $N > (d+2C_\omega)/\lambda_0$,

$$
\begin{aligned}
\omega(Q > N) \;\;\leqslant\;\; & \sum_{\nu \in \mathbb{Z}^d} \sum_{k > \log^{1/d}(e+|\nu|)} e^{-N\lambda_0(2k+1)^d/2} \\
\leqslant\;\; & C \sum_{\nu \in \mathbb{Z}^d} \frac{1}{(e+|\nu|)^{N\lambda_0}} \leqslant C e^{-N\lambda_0},
\end{aligned}
$$

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which implies (2.6) for any $\lambda < \lambda_0$. ∎

Proof of Theorem 2.1. For any initial condition $x = \{x_i\}_{i \in \mathbb{Z}^d} \in \mathcal{X}_0$, the solution to Eq. (2.1) is constructed as the limit

$$\Phi_t(x)_i = \lim_{n \to \infty} \Phi_t^{(n)}(x)_i \qquad \forall\, i \in \mathbb{Z}^d, \tag{A.1}$$

where the *n-partial dynamics* $\Phi_t^{(n)}(x)$ is defined in the following way. For any $n \in \mathbb{N}$ let $\Lambda_n = \Lambda_{0,n}$ be the cube of side $2n+1$ centered in the origin. Then $\Phi_t^{(n)}(x) = \{(q_i^{(n)}(t), p_i^{(n)}(t))\}_{i \in \Lambda_n}$ is the solution to the Cauchy problem

$$\begin{cases} \dot{q}_i^{(n)}(t) = p_i^{(n)}(t), \\ \dot{p}_i^{(n)}(t) = F_i(\Phi_t^{(n)}(x)), \\ (q_i^{(n)}(0), p_i^{(n)}(0)) = x_i, & i \in \Lambda_n, \end{cases} \tag{A.2}$$

where

$$F_i(\Phi_t^{(n)}(x)) = -U'(q_i^{(n)}(t)) - K \sum_{j \in \Lambda_n : |j-i|=1} \left[q_i^{(n)}(t) - q_j^{(n)}(t) \right]. \tag{A.3}$$

Lemma 3.2 There exists a positive constant C_2 such that, for any $x \in \mathcal{X}_0$, $t \in \mathbb{R}$, $n \in \mathbb{N}$, and $\Lambda_{\nu,k} \subseteq \Lambda_n$,

$$W_{\nu,k}(\Phi_t^{(n)}(x)) \leqslant C_2 \{ Q(x) [\log(e+n) + k^d] + t^4 \}, \tag{A.4}$$

where $W_{\nu,k}(\cdot)$ is defined in (2.3).

Proof. For notational simplicity we consider the case $t > 0$. By the equations of motion,

$$\frac{d}{dt} W_{\nu,k}(\Phi_t^{(n)}(x)) = -K \sum{}^* \left[q_i^{(n)}(t) - q_j^{(n)}(t) \right] p_i^{(n)}(t),$$

where $\sum^*$ denotes the sum over all the sites $i \in \Lambda_{\nu,k}$ and $j \in \Lambda_n \setminus \Lambda_{\nu,k}$ such that $|i - j| = 1$. Recalling the assumptions on the one body potential $U(\cdot)$, we have

$$\begin{aligned} \left| \frac{d}{dt} W_{\nu,k}(\Phi_t^{(n)}(x)) \right| &\leqslant C \sum_{i \in \Lambda_{\nu,k}} |p_i^{(n)}(t)| \sum_{j:|j-i|\leqslant 1} |q_j^{(n)}(t)| \\ &\leqslant C \left[W_{\nu,k}(\Phi_t^{(n)}(x)) \right]^{1/2} \left[W_{\nu,k+1}(\Phi_t^{(n)}(x)) \right]^{1/4}. \end{aligned}$$

On the other hand, $W_{\nu,k+1} \leqslant \sum'_{\nu'} W_{\nu',k}$, where $\sum'_{\nu'}$ denotes the sum over all the sites ν' such that $|\nu_\ell - \nu'_\ell| \leqslant 1$ for some $\ell = 1, \dots, d$; then, letting

$$\overline{W}_k(t) = \max_{\nu:\Lambda_{\nu,k} \subseteq \Lambda_n} W_{\nu,k}(\Phi_t^{(n)}(x)),$$

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we get

$$\left|\frac{d}{dt}W_{\nu,k}\big(\Phi_t^{(n)}(x)\big)\right| \leqslant \overline{W}_k(t)^{3/4}.$$

Now, by integrating the last inequality from 0 to t and taking the maximum for ν such that $\Lambda_{\nu,k} \subseteq \Lambda_n$,

$$\overline{W}_k(t) \leqslant \overline{W}_k(0) + C \int_0^t ds\, \overline{W}_k(s)^{3/4},$$

whence $\overline{W}_k(t) \leqslant \big(\overline{W}_k(0)^{1/4} + Ct\big)^4$. But from the definition (2.4) we have that $\overline{W}_k(0) \leqslant Q(x)\big\{2\big[\log^{1/d}(e+n) + k\big] + 1\big\}^d$. The bound (A.4) follows from the previous estimates. ∎

We now prove the existence of the limit (A.1). Again we consider the case $t > 0$. We define:

$$
\begin{aligned}
\delta_i(n,t) &= \big|q_i^{(n+1)}(t) - q_i^{(n)}(t)\big| + \big|p_i^{(n+1)}(t) - p_i^{(n)}(t)\big|, \\
u_k(n,t) &= \max_{i \in \Lambda_k} \delta_i(n,t), \qquad k \leqslant n, \\
d_n(t) &= \max_{s \in [0,t]} \max_{i \in \Lambda_n} \big\{\big|q_i^{(n)}(s) - q_i^{(n)}(0)\big| + \big|p_i^{(n)}(s) - p_i^{(n)}(0)\big|\big\}.
\end{aligned}
$$

From (A.2) and (A.3), for any $i \in \Lambda_k$, we have

$$
\begin{aligned}
\delta_i(n,t) &\leqslant \int_0^t ds\, \big|p_i^{(n+1)}(s) - p_i^{(n)}(s)\big| \\
&\quad + \int_0^t ds\, \big|F_i(\Phi_s^{(n+1)}(x)) - F_i(\Phi_s^{(n)}(x))\big| \\
&\leqslant \int_0^t ds\, \big[1 + 2dK + \big|U''(\xi_i^n(s))\big|\big]\, u_{k+1}(n,s),
\end{aligned}
$$

where $\xi_i^n(s)$ is a point in the interval with endpoints $q_i^{(n)}(s)$ and $q_i^{(n+1)}(s)$. By (A.4), for any $s \in [0,t]$,

$$\big|U''(\xi_i^n(s))\big| \leqslant C\big[1 + q_i^{(n)}(s)^2 + q_i^{(n+1)}(s)^2\big] \leqslant C\big\{Q(x)\log(e+n) + t^4\big\}^{1/2},$$

so that, setting $\varphi_n(t,x) = \big\{Q(x)\log(e+n) + t^4\big\}$ and taking the maximum for $i \in \Lambda_k$, we arrive at the following integral inequality,

$$u_k(n,t) \leqslant C\,\varphi_n(t,x)^{1/2} \int_0^t ds\, u_{k+1}(n,s),$$

which can be solved by iteration getting, for any $n > k$,

$$u_k(n,t) \leqslant C^{n-k}\,\varphi_n(t,x)^{(n-k)/2}\, \frac{t^{n-k}}{(n-k)!}\, d_n(t).$$

10

By using the Stirling formula and observing that

$$d_n(t) \leqslant t \max_{s \in [0,t]} \max_{i \in \Lambda_n} \left\{ \left| p_i^{(n)}(s) \right| + \left| F_i(\Phi_s^{(n)}(x)) \right| \right\} \leqslant C\, t\, \varphi_n(t,x),$$

we finally obtain

$$u_k(n,t) \leqslant \frac{C^{n-k}\, t^{n-k+1}\, \varphi_n(t,x)^{(n-k+2)/2}}{(n-k)^{n-k}}.$$

Choosing $n_k = n_k(t,x) = 2k + C_3 \left[1 + Q(x) + t \right]^4$ with C_3 large enough, it is easily seen that $u_k(n,t) \leqslant 2^{-(n-k)}$ for any $n \geqslant n_k$. It follows that $u_k(n,t)$ is k-summable, which implies the existence of the limit (A.1). Indeed, we have also an estimate on the rate of convergence: for some constant $\delta > 0$,

$$\left| \Phi_t(x)_i - \Phi_t^{(n_k)}(x)_i \right| \leqslant e^{-\delta n_k} \qquad \forall i \in \Lambda_k. \tag{A.5}$$

The proof of the differentiability of $\Phi_t(x)_j$ with respect to x_i is quite standard and we omit the details. Note however that we explicitly solved the corresponding variational equation (for $i = 0$), see Eqs. (3.7)–(3.9).

Finally, we prove the estimate (2.7). Obviously, this also shows that $\Phi_t(x) \in \mathcal{X}_0$ for any $t \in \mathbb{R}$. Given $\nu \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, we set $n_* = n_{|\nu|+k}(t,x)$ and estimate

$$W_{\nu,k}\big(\Phi_t(x)\big) \leqslant W_{\nu,k}\big(\Phi_t^{(n_*)}(x)\big) + \left| W_{\nu,k}\big(\Phi_t(x)\big) - W_{\nu,k}\big(\Phi_t^{(n_*)}(x)\big) \right|. \tag{A.6}$$

By (A.4), for any $k > \log^{1/d}(e + |\nu|)$,

$$\begin{aligned}
W_{\nu,k}\big(\Phi_t^{(n_*)}(x)\big) &\leqslant\ C_2\big\{ Q(x)\big[\log(e + n_*) + k^d \big] + t^4 \big\} \\
&\leqslant\ C\, (2k+1)^d \big\{ Q(x) \log(e + Q(x)) + t^4 \big\}. \tag{A.7}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Big| W_{\nu,k}\big(\Phi_t(x)\big) - W_{\nu,k}\big(\Phi_t^{(n_*)}(x)\big) \Big| &\leqslant\ (2k+1)^d \\
&\times \max_{i \in \Lambda_{\nu,k}} \left\{ \frac{1}{2} \left| p_i(t)^2 - p_i^{(n_*)}(t)^2 \right| + \left| U(q_i(t)) - U(q_i^{(n_*)}(t)) \right| \right\} \\
&\leqslant\ C\,(2k+1)^d\, \varphi_{n_*}(t,x)^{3/4} \max_{i \in \Lambda_{\nu,k}} \left| \Phi_t(x)_i - \Phi_t^{(n_*)}(x)_i \right| \\
&\leqslant\ C\,(2k+1)^d\, \varphi_{n_*}(t,x)^{3/4}\, e^{-\delta n_*} \leqslant C, \tag{A.8}
\end{aligned}$$

where we used (A.5). From (A.6), (A.7), and (A.8) the bound (2.7) follows. ∎

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